Travis Schedler

Geometric Representation Theory.

Idea: Groups acting on spaces, categories... objects.

SYMMETRIES — in different forms.

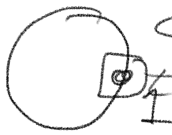
$\to$ also include: Lie algebras. (infinitesimal symmetries)

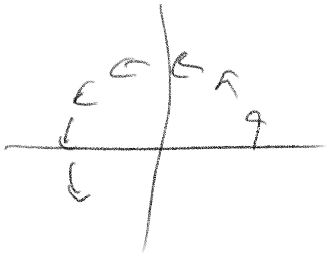
Eg: $S^1 \circlearrowright \mathbb{R}^n$ rotation

$\cup$

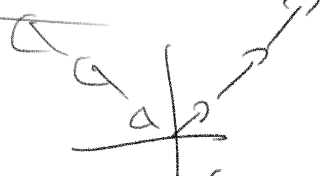
$C_n \sim \frac{2\pi k}{n}$ rotations  rot^n

group acts by automorphisms

Lie alg:  S^1, I — zoom in: $\mathbb{R} = 1D$ v.s.

acts on $\mathbb{R}^2$ by: 

rotational vector field: $y\partial_x - x\partial_y$

Radial: $x\partial_x + y\partial_y$ 

exponentiate / integrate: recovers global symmetry.

More exotic symmetries:

- quantum groups / Hopf algebras
- operator algs / differential ops.

Main idea of geometric Rep theory:

Classical Rep Theory. Algebra (groups, Lie algs) $\xrightarrow[\text{symmetry}]{\text{classify}}$ Symmetry of geometry, physics

Geom. Rep Theory. Construct Representations $\rightsquigarrow$ rich geometric spaces (Flag varieties, $\mathbb{P}^n$, homogeneous spaces...) AG / physics / SG

Defn: A representation is:

of group: $G \xrightarrow{\text{homom.}} \mathrm{Aut}(X) = GL(V)$

usually. X = vector space: "linear rep"

X = linear object.

of Lie alg: $\mathfrak{g} \xrightarrow[\text{of Lie algs.}]{\text{homom}} \mathfrak{gl}(V)$

Lie alg: G = Lie or algebraic group (manifold, group str.)

$T_e G =: \mathfrak{g}$, $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$

or: $\mathrm{Vect}(G)^G$

skew symm, bilinear, Jacobi identity.

Flavour of GRT:

· Borel-Weil + Beilinson-Bernstein:

G = semisimple Lie or algebraic group

(essentially: product of simple: no normal subgp)

ex: $SL_n = \{A \in GL_n \mid \det A = 1\}$

$SO_n = \{ \quad " \quad \mid \det A = 1, \ AA^t = I\}$

$Sp_{2n} = \{A \in GL_{2n} \mid \omega(Av, Aw) = (v, w)\}$

$B < G$ Borel subgroup (max solvable)

G/B =: "flag variety"

Homogeneous action of G.

$= \{ \mathfrak{b} \subseteq \mathfrak{g} \mid \mathfrak{b} \text{ max solvable}\}$

$\mathfrak{g} = \text{Lie } G$

$N_G(\mathfrak{b}) = B$

+ All $\mathfrak{b}$ are conjugate

Case $G = SL_n$: Standard $B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}$, $\det = 1$

solvable.

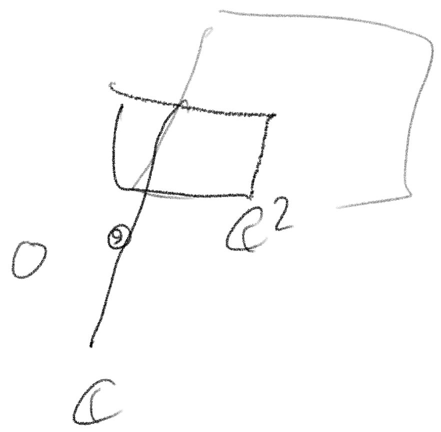
work over $\mathbb{C}$

B preserves flag:

$$0 \subseteq \begin{pmatrix} * \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \subseteq \begin{pmatrix} * \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix} \subseteq \cdots \subseteq \begin{pmatrix} * \\ \vdots \\ * \end{pmatrix} = \mathbb{C}^n$$

$\langle e_1 \rangle \cong \mathbb{C}$, $\quad \langle e_1, e_2 \rangle \cong \mathbb{C}^2$, $\quad \ldots = \langle e_1, \ldots, e_n \rangle$

Ex: $A \in B \iff A$ preserves each space in flag



$\mathfrak{g} = \text{Lie}\, G$

G semisimple (e SL_n)

$\mathfrak{sl}_n = \{A \in \mathfrak{gl}_n \mid \operatorname{tr} A = 0\}$
$\qquad\qquad\qquad\quad \text{Mat}_{n\times n}$

Borel-Weil: Every irreducible f.d. rep of G (or $\mathfrak{g}$) is the global sections of line bundle on G/B.
[+ conversely]

Generally:

$G \circlearrowright X$ ~ geom space, nonlinear

$\Rightarrow G \circlearrowright \Gamma(X, \mathcal{O}_X) :=$ global functions

Here: X, L = line bundle (algebraic / holom.) (equivariant)

$\Rightarrow G \circlearrowright \Gamma(X, L)$

Ex: $G = SL_2 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1\}$

$\mathfrak{g} = \mathfrak{sl}_2 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + d = 0\}$

$G/B \cong \{\text{lines in } \mathbb{C}^2\} = \mathbb{P}^1$

$\Rightarrow$ line bundles on $\mathbb{P}^1$ (alg / holom.)

- taut. "$\mathcal{O}(-1)$"

- $\mathcal{O}(n)$, $n \in \mathbb{Z}$

$\mathrm{Pic}\,\mathbb{P}^1 \cong \mathbb{Z}$

"line bdles/$\cong$"

Irreps of SL_2 $\Leftrightarrow$ f.-d. irreps of $\mathfrak{sl}_2$ $= \{ \mathbb{C}[x,y]_k$ degree k polys in $x,y \}$ $= \Gamma(\mathbb{P}^1, \mathcal{O}(k))$

$\Updownarrow$

$k \in \mathbb{Z}_{\geq 0}$

dim: $k+1$

Borel-Weil-Bott: extends from Γ to $H^i(G/B, L)$

$\Rightarrow$ get $H^i(\mathbb{P}^1, \mathcal{O}(m)) = \begin{cases} \mathbb{C}[x,y]_m, & m \geq 0,\ i=0 \\ \mathbb{C}[x,y]_{-2-m}, & m \leq -2,\ i=1 \\ 0 & \text{otherwise} \end{cases}$

Beilinson–Bernstein:

generalisation to <u>infinite-dim</u> reps of $\mathfrak{g}$ (semisimple)

twisted $\mathcal{D}$-modules on G/B "systems of alg diff eqns" $\xleftrightarrow{1-1}$ Reps of $\mathfrak{g}$ (all!)

2D singularities / Platonic solids.

governed by symmetry groups

$S^3 \subseteq \mathbb{H}$

$\cong$

$Spin(3,\mathbb{R}) \xrightarrow{2:1} SO(3,\mathbb{R})$

Rotations of $\mathbb{R}^3$:

not simply-conn. $\pi_1 = C_2$

$\cong$

$SU(2,\mathbb{C})$

$\det A = 1$, $A = \begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix}$, $|a|^2+|b|^2=1$

$A\bar{A}^t = I$

$GL(V)$

fd irr. Reps of $SU(2,\mathbb{C}) \longleftrightarrow$ fd irreps of $SL(2,\mathbb{C})$

$SU(2,\mathbb{C}) \underset{\text{max cpt.}}{\leq} SL(2,\mathbb{C})$

Finite $G < SU(2,\mathbb{C}) \longleftrightarrow$ finite $G < SL(2,\mathbb{C})$

up to conjugate

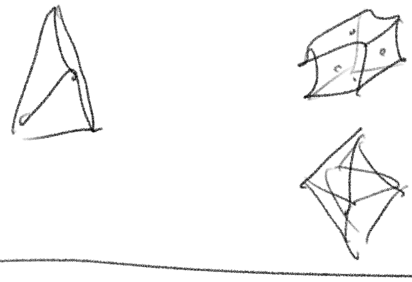
$\rightsquigarrow$ Classification of finite $G < SU(2,\mathbb{C})$

$\Leftrightarrow G < SL(2,\mathbb{C})$

- cyclic $\cong \left(\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{smallmatrix}\right)$, $\zeta^n = 1$
- binary dihedral $\cong \langle C_n, \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$
- binary tetrahedral, octahedral, icosahedral

?

rotational symms of



icosahedron

dodecahedron.

Given $G < SL(2,\mathbb{C})$ or $SU(2,\mathbb{C})$ $\rightsquigarrow$ $\mathbb{C}^2/G$

<u>quotient sings</u>.

<u>Ex</u>: $G = C_2 = \{\pm I\}$

$\mathbb{C}^2/\{\pm I\} = \{\{\pm z\}\}$

Functions on it: <u>even</u> functions on $\mathbb{C}^2$

$$\mathbb{C}[x,y]^{C_2} = \mathbb{C}[x^2, xy, y^2]$$
$$\shortparallel$$
$$\mathbb{C}[u,v,w]/(uw - v^2 = 0)$$

A_1 Du Val sing.

$\mathbb{C}^2/\{\pm I\}$  $\subseteq \mathbb{C}^3$

$uw - v^2 = 0$

$\cong$

 $= \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| a^2 + bc = 0 \right\} \subseteq \mathfrak{sl}_2 \cong \mathbb{C}^3$

$\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\}$

$= \{A \mid A^2 = 0\}$

$\mathrm{Nil}_{2\times 2}$

McKay Corr:

Dynkin
Cartan-Killing:
early 20th c.

Simple $\mathbb{C}$ Lie alg's

$\updownarrow$

Diagrams ADE

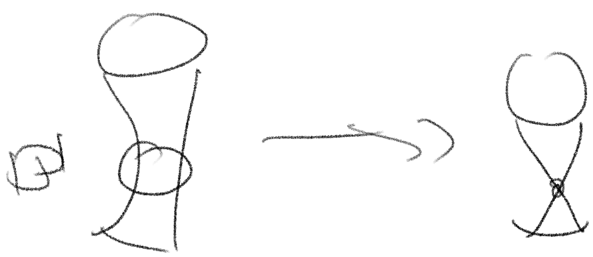
A $\qquad\qquad$ D

E_6 $\qquad$ E_7 $\qquad$ E_8

These graphs are: zero fiber

I) adjacency of minimal resolution of $\mathbb{C}^2/G$

Ex:



$$T^*\mathbb{P}^1 \longrightarrow\!\!\!\!\rightarrow \mathrm{Nil}_{2\times 2}$$

$= \{ A \in \mathrm{Nil} \mid \ell \subseteq \mathbb{C}^2 \text{ line}, \ \ell \subseteq \ker A \}$

Get graph $\bullet$ $(\mathfrak{sl}_2)$

General:

A_2 $\longleftrightarrow$ $\mathfrak{sl}_3$.

II $\quad G \leq SU(2,\mathbb{C}) \rightsquigarrow$ Dynkin diags $\rightsquigarrow$ Lie alg

vertices = nontriv irreps of G

edges: ρ — ρ' if $\rho \subseteq \rho' \otimes \mathbb{C}^2$

$\mathbb{C}^2/\Gamma \xrightarrow{\text{McKay graph of } \Gamma}$ Dynkin $\rightsquigarrow$ Lie algs

{Du Val sings}

$\widetilde{\mathbb{C}^2/\Gamma}$ $\rightsquigarrow$ adjacency of $f^{-1}(0)$, f = resol.

"Springer resol" $T^*G/B \to Nil(\mathfrak{g})$